THE MATHEMATICAL STRUCTURE OF CLASSICAL MECHANICS

Abstract. Newton's laws of motion do not allow us to identify an inertial frame of reference. To find such a frame we must identify either an unaccelerated body, or a body on which no force acts. The first approach is rendered impossible by the relativity of motion, while the second requires an untestable theory of force. We present an axiomatization of classical mechanics that avoids these problems. Our approach is operational. First, we give an axiomatization of clocks and laboratories. Second, we derive from this axiomatization the properties of spacetime. We then show how the motions of a system of bodies may be described in this formalism.

1. Introduction

This paper is the result of a discussion between a physicist and a mathematician. We were dissatisfied with elementary presentations of classical mechanics as given in graduate and undergraduate textbooks, especially so with the sections of those presentations that deal with inertial reference frames and the concept of force. We asked ourselves, 'How is classical mechanics properly formalized? How do inertial reference frames and force fit into this formalization?' Classical mechanics studies the *motion* of physical bodies, namely the changes in a physical body's position over time. Indeed James Clerk Maxwell [1] called his primer on classical mechanics 'Matter and motion'. Classical mechanics is a quantitative science, and puts values on these motions. It concerns *measurements* of motions. Specifically, it assigns real numbers to the positions of a body, and also to the times at which a body has these positions (and hence its velocity and acceleration, etc.). Classical mechanics therefore quantifies *time* and *space*. We first note that motion is an inherently *relative* phenomenon: the motion of a body is necessarily defined with respect to an *observer*, that is to say it requires a *reference frame*.

Hold out your finger. Right now, the tip of your finger marks a point in space and time. Now wait a moment. It no longer marks the same point in time, but does it now mark the *same* point in space? Relative to you and the room in which you are standing, it does. But consider the fact that Earth is spinning about an axis. For an observer at one of the poles, it traces out a circle. Of course Earth is also orbiting the Sun. For an observer at the Sun, your finger moves along an epicycle. The Sun is in turn orbiting the Galactic centre, which is moving towards Andromeda, and so on. This regression does not terminate, so that there is no observer whom we may single out as having a special status. Any two points that you identify as the same may be identified as different by another observer. As no independent observer exists to arbitrate we cannot say the you are right, and we conclude that there is no sense in which points in space persist over time. Using the same argument we see that there are no absolute motions in spacetime. For example, there is no sense in which a line can be straight, independent of an observer.

Classical mechanics also makes *predictions* about the motion of physical bodies. Given sufficient knowledge of a body's current state, classical mechanics predicts its future state (or, indeed, postdicts its past state). These motions will be different in two arbitrarily chosen reference frames, giving us the problem that our predictions must apply to all of them. Consider the treatment of this problem given by French and Ebison [2], which has much in common with other undergraduate textbooks. They identify a special class of reference frames (named *inertial*), and then state the laws of motion in a form that describes motions in these frames. (We may then describe motions in any other frame by making the appropriate transformation.) Their presentation follows that of Newton, insofar as it invokes his three laws. The difficulty here is that, contrary to our previous observations, Newton's presentation does in fact have a concept of absolute motion. Thus, modern statements of Newton's laws have a meaning different from the original.

Let us, then, recapitulate Newton's account of mechanics. First (and contrary to the modern understanding we have just outlined) Newton assumes the existence of an absolute (i.e. distinguished) reference frame, with respect to which we may measure absolute accelerations. He considers force to be a primitive quantity (i.e. an intuitive concept that needs no formal definition) and says that forces *act* on bodies, or that bodies are *acted upon* by forces. He also considers mass to be a primitive quantity and associates a mass with every body. Recall that Newton uses the term 'quantity of motion' to refer to momentum. In its original form [3] Newton's axiomatization of mechanics is as follows.

N1 (Newton's first law).: Every body continues in its state of rest, or of uniform motion in a right line, unless it is compelled to change that state by forces impressed upon it.

N2 (Newton's second law).: The change of motion is proportional to the motive force impressed; and is made in the direction of the right line in which that force is impressed.

N3 (Newton's third law).: To every action there is always opposed an equal reaction: or the mutual actions of two bodies upon each other are always equal, and directed to contrary parts.

The first law states that a body has zero absolute acceleration if no force acts on it, whereas the second law states that a force causes the absolute acceleration of a body. Taken together they state that a body undergoes absolute acceleration if and only if a force acts on it. Without these stipulations, a body might accelerate (i.e. accelerate absolutely) without a force acting on it, or a force might act on a body but not accelerate it (i.e. not accelerate it absolutely). The third law states that a force is the result of an interaction between a pair of bodies, and that this interaction is antisymmetric. The first part of the third law (concerning interactions) rules out background forces. Without it, background forces might act. The second part of the third law (concerning antisymmetry) states that all bodies are sources of force. Without it, a body might be a source of force but not be acted upon by forces (or it might be acted upon by forces but not be a source of force).

Without the concept of an absolute reference frame, modern statements of Newton's laws differ from the original. French and Ebison [2], for example, put it as follows.

G1.: There exist certain frames of reference with respect to which the motion of an object, free of all external forces, is a motion in a straight line at constant velocity (including zero).

G2.: Newton's second law gives us a quantitative relationship:

$$\mathbf{F} = m\mathbf{a} = m\frac{\mathrm{d}\mathbf{v}}{\mathrm{d}t}$$

where the proportionality factor m is called the *inertial mass* of the object and $\mathbf{F}$ is the net force acting on it.

G3.: The total momentum of a system of two colliding particles remains unchanged by the collision, i.e. the total linear momentum is a conserved quantity.

From G3 French and Ebison then derive the fact that for two interacting bodies, i and j, if a force $\mathbf{F}_{ij}$ acts on body i then a force $\mathbf{F}_{ji}$ acts on body j and that $\mathbf{F}_{ij} = -\mathbf{F}_{ji}$. (French and Ebison's reference to *colliding* bodies is in fact a reference to *interacting* bodies.) Moreover, the total force $\mathbf{F}_i$ on a body i is the sum of the forces $\mathbf{F}_{ij}$ exerted on that body by each other body j.

Rather than stating that there exists an absolute frame in which accelerations are caused only by force, G1 states that there exists a class of frames in which relative accelerations are caused only by forces. Note that force is still a primitive quantity. A force is to be considered as being *physical*, and not *fictitious* (i.e. not a result of the observer's motion). We call the equation given in G2 *Newton's equation*. Newton's equation holds in an inertial frame.

But how do we find an inertial frame? Statement G1 does not give a prescription for finding one, yet we *must* find one in order to test the claims of G2 and G3. Only one course of action is available to us: we must identify a body on which zero force acts. Any frame in which such an isolated body moves uniformly is then an inertial frame. The laws themselves are of limited help here. From them we know only that zero force acts on a body if we observe it moving uniformly in an inertial frame. To use this fact alone would have us descend into circularity. We would know that we are in an inertial frame because no force acts on a uniformly moving body. Yet we would only know that no force acts on that body because we observe uniform motion in an inertial reference frame. We need some other means of identifying a body on which zero force acts. In this respect we need some minimal *theory* of force. The normal way of fulfilling this requirement is described by Kibble and Berkshire [4]. We assume that on cosmological scales only gravitational forces act, and that the gravitational force acting on one body due to another decreases monotonically to zero as their separation tends to infinity. Thus, a body separated from all others by some great distance may be considered isolated. The weakness of this approach is that it requires untestable assumptions. (Kibble and Berkshire of course acknowledge this problem.) The situation is unsatisfactory. Without a theory of force we are doomed to circularity in our attempt to identify an inertial reference frame. With one, we are doomed to make untestable assumptions.[1]

We propose that the solution to the problem is to explicitly state the equations of motion that describe a physical system, without reference to a special class of frames. For example, it is sufficient to state that a gravitational system of n bodies is, by definition, a system for which the acceleration of the ith body is

$$\ddot{\mathbf{x}}_i(t) = \sum_{j\neq i} \frac{m_j(\mathbf{x}_j(t) - \mathbf{x}_i(t))}{||\mathbf{x}_j(t) - \mathbf{x}_i(t)||^3}.$$

In an arbitrary frame, moving with acceleration $\ddot{\mathbf{X}}(t)$ and angular velocity $\mathbf{\Omega}(t)$, the right-hand side of this equation is augmented by additional terms, giving us the acceleration

$$\ddot{\mathbf{x}}_i(t) = \sum_{j\neq i} \frac{m_j(\mathbf{x}_j(t) - \mathbf{x}_i(t))}{||\mathbf{x}_j(t) - \mathbf{x}_i(t)||^3} - \ddot{\mathbf{X}}(t) - \mathbf{\Omega}(t) \times (\mathbf{\Omega}(t) \times \mathbf{x}_i(t)) \\ -2\mathbf{\Omega}(t) \times \dot{\mathbf{x}}_i(t) - \dot{\mathbf{\Omega}}(t) \times \mathbf{x}_i(t).$$

Crucially this statement is testable. It requires that we make a hypothesis about the functions $\ddot{\mathbf{X}}$ and $\mathbf{\Omega}$, but this is not problematic. It simply means that we are testing a claim about the existence of a particular gravitational system *alongside* a claim about the motion of a particular observer. The functions $\ddot{\mathbf{X}}$ and $\mathbf{\Omega}$ are arbitrary (notwithstanding the fact that $\mathbf{x}_j$ must be twice differentiable in any frame) and we must either find them informally or make approximations to them. For example, we might suppose that over a given time scale they are approximately constant. Gravitational systems, in the sense defined above, may not exist (in point of fact, they do not, as we know classical mechanics to be an inadequate theory of mechanics) and such a statement allows for this possibility. Note that, in this formalism, force is no longer a primitive quantity. Instead, we consider accelerations directly.

Thus far we have assumed that quantities of time are elements of a one-dimensional normed real vector space, and that quantities of space are elements of a three-dimensional normed real vector space. Further, we have assumed that these quantities of space are twice differentiable. We might suppose that these properties reflect the structure of some underlying spacetime. For example, Arnold [5] defines spacetime to be a four-dimensional affine space on which the four-dimensional vector space V acts as *the group of parallel displacements*. He defines time to be a surjective linear mapping, $t : V \longrightarrow \mathbb{R}$, from spacetime to the real numbers, the kernel of which is a three-dimensional vector space, U, cosets of which, $[a]$, are equivalence classes of simultaneous events. The kernel is equipped with a real-valued scalar product, $[\cdot,\cdot] : U \times U \longrightarrow \mathbb{R}$, which induces a metric, $\rho : [a] \times [a] \longrightarrow \mathbb{R}$ on each of these equivalence classes. We might call this the *geometric approach* to the axiomatization of classical mechanics. A similar approach is taken by Penrose [6].

[1] Moreover, we need a full theory of force to test the claim of G2, for to test the claim that acceleration is proportional to force we must be able to quantify that force, just as we can quantify the acceleration.

However, there are two problems with the geometric approach. First, affine four-space carries with it a notion of straight lines. Yet we have already seen that there are no straight lines independent of an observer. Second, the codomain of both the scalar product and the time mapping is the set of real numbers, meaning that they each come with a distinguished element, one. Yet neither space nor time has a canonical unit. It is also the case the geometric approach does not explicitly treat the measurements an observer may make of spacetime. Presumably, the value $\rho(a, b)$ corresponds to the distance between two elements of spacetime, a and b, as measured by some rod at a given time. Similarly, the value $t(b-a)$ presumably corresponds to the time difference between two elements of spacetime, a and b, as measured by a clock. But in order to make make such measurements we need to know which instruments are, in fact, rods and clocks. In the geometric approach, a structure is imposed on spacetime *a priori*, and the possible measurements of spacetime come later, if at all.

It is possible to take the opposite approach, first defining the instruments we use to make measurements of time and space, and from these deriving a structure for spacetime. We might call this approach *operational* or *instrumental*. This approach was of course pioneered by Einstein in his treatment of relativistic mechanics. In what follows we provide an axiomatization of classical mechanics according to this instrumental approach. The distinguishing features of this axiomatization are that we take the notions of events, observers and measurements to be primitive, that we give formal definitions of the clocks and rods that an an observer must use to make these measurements, and that we do so in an entirely scale-free manner. We do not build into our account of spacetime a special class of frames, as is done in the geometric approach. Instead, we consider arbitrary frames, noting that the most that is possible is for a pair of frames to be *relatively inertial*, such that a uniform motion in one frame is also uniform in the other. Having established an axiomatization of spacetime we then reprise our treatment of physical systems in a way that avoids the circularity and untestability of conventional treatments.

2. Observers and events

We will denote the set of *observers* by O, and the set of *events* by S. We will call S *spacetime*. An observer may *measure* the *time* at which an event occurs using a *clock*, and may similarly measure the *position* at which an event occurs using a system of rods, which we will call a *laboratory*.

2.1. Clocks and laboratories.

Clocks and laboratories. Clocks record the time that that has elapsed between two events by measuring some regular process: the burning of a candle, the draining of sand from a bulb, the oscillating of a pendulum. But we may ask, with respect to what is a process regular? If an observer has only one clock there is no meaning to the notion of regularity. A single clock may not be said to keep regular time. But two clocks together may be said to keep regular time with respect to each other. Rather than consider a prescription for making a clock, we consider the properties that a clock—or rather, system of clocks—must have. Let us consider a tape machine, which consists of three parts: an arbitrarily long piece of tape, a feeding mechanism, and a print head that makes marks on that piece of tape. We may identify a given event in spacetime with a mark on the tape. What properties must the machine have if that mark is to represent the time at which the event happened? That is to say, what properties must the machine have if it is to constitute a clock? We cannot say that the feeding mechanism moves the tape regularly, only that it allows the print head to make *some* set of distinct marks on the tape. Nor can we say that the feeding mechanism moves the tape *forwards* or *backwards*. It may run the paper in either direction. Nonetheless the machine does have a notion of one point's being *between* two others.

We may think of this tape machine as doing one of two equivalent things. On the one hand it allows the observer o to measure the ratio of time intervals defined by quadruples of events. Consider four events $a, b, c, d \in S$. Then the ratio of the time interval between a and b to the time interval between c and d is a real number (i.e. it may be positive, negative, or zero) or undefined. In this sense a clock is a function

$$\tau_o : S^4 \longrightarrow \mathbb{R} \cup \{\infty\}$$

from the set of quadruples of events to the union of the set of real numbers and infinity.[2] On the other hand, this machine allows the observer o to measure the time interval between two events, but not as a real number. Consider two events $a, b \in S$. If the machine makes two marks on its tape corresponding to a and b then these marks represent the time interval between a and b. But this time interval is not a real number. Instead, it is an element of a vector space. In this sense a clock is a function

$$t_o : S \longrightarrow T$$

from the set of events to an affine line, i.e. a one-dimensional real affine space.

Physical laboratories measure the position and orientation of an event by means of some rigid, oriented framework of rods. But we may ask, with respect to what is this framework rigid and oriented? Again, we do not consider a prescription for making a laboratory. Rather, we consider the properties that a laboratory—or rather, a system of laboratories—must have. A convenient image is of a system of rods, extending radially from an observer, together with a second system of rods placed such that each exactly separates the tips of each pair of rods in the first system.[3] The position of each event in spacetime is marked by a rod from the first system. Indeed each such a rod marks a different event at each instant of time. A rod from the second system thus represents the separation of both simultaneous and non-simultaneous events. Again, we may think of a laboratory as doing one of two things. On the one hand it allows the observer o to measure the ratio of distances defined by quadruples of (not-necessarily simultaneous) events. Consider four events $a, b, c, d \in S$. Then the ratio of the distance between a and b to the distance between c and d is a nonnegative real number or undefined. In this sense a laboratory is a function

$$\delta_o : S^4 \longrightarrow \mathbb{R}_{\geq 0} \cup \{\infty\}.$$

On the other hand, a laboratory allows the observer o to measure the distance between two (not-necessarily simultaneous) events but, again, not as a real number. Instead, such a distance is an element of a half-line, i.e. a choice of one half of a one-dimensional real vector space (see appendix for a full discussion of half-lines). In this sense a laboratory is a function

$$r_o : S \longrightarrow D_o$$

from spacetime to a three-dimensional real vector space equipped with a norm $\|\cdot\| : D_o \longrightarrow H_o$, the codomain of which is a half-line H_o.

Clocks and laboratories are consistent in the following sense. Any two observers agree on the ratio of time intervals defined by a quadruple of events. Any two observers agree on the ratio of distances defined by two pairs of *simultaneous* events. They also agree on the preservation of orientations over time. Consider a quadruple of simultaneous events, a, b, c, and d, such that no three events are collinear. Events a, b, and c define a plane, meaning that d may be said by an observer to lie on one side of it. This defines an orientation of the quadruple. Now consider two such quadruples of simultaneous events. If one observer considers the two quadruples to have the same orientation, then so does another observer. Note that two observers can always agree that no three points are collinear as they agree on ratios of distances defined by quadruples of simultaneous events.

We emphasize that our intention here has not been to give a prescription for building clocks and laboratories. Rather we have given necessary and sufficient conditions for a given system

[2]We choose to formalize a clock τ (or τ_o) to have codomain including the *symbol* ∞. Under the clock axioms, by Theorem 3, τ will correspond to a map t, and $\tau(a, b, c, d) = \infty$ will correspond to $t(c) = t(d)$, so that the ratio $(t(a) - t(b))/(t(c) - t(d))$ is undefined. Later on, we will use the same convention for the maps δ and r.

[3]It will be the case that laboratories define a three-dimensional space. But we do not speak of a lattice of rods, as we wish to avoid the idea that we are working with a specific basis.

to be considered a clock or laboratory. In practice, we find that certain oscillators are indeed clocks, and that certain systems of rods are indeed laboratories.

3. Spacetime

We now formalize our description of clocks and laboratories and use this formalization to deduce a structure for spacetime. In seeking a formalization of spacetime (and of classical mechanics, which we deal with later) we take for granted that we are seeking a *minimal* formalization, i.e. we are seeking the minimal mathematical structure required to adequately describe classical mechanics.

3.1. **Time.**

T1 (measurability).: Each observer $o \in O$ is equipped with a *clock*.

This means that there is a map

$$\tau_o : S^4 \longrightarrow \mathbb{R} \cup \{\infty\},$$

called the *ratio of time differences*, and which satisfies the *clock axioms* below.

T2 (consistency).: Observers agree on the ratio of time differences between quadruples of events.

That is, for any $o, o' \in O$ we have $\tau_o = \tau_{o'}$. Thus, for $a, b, c, d \in S$ we have $\tau_o(a, b, c, d) = \tau_{o'}(a, b, c, d)$. Therefore there is really only one ratio of time-differences map, which will be denoted τ.

Definition 1. We say that events $a, b \in S$ are *simultaneous*, and write $a \sim_\tau b$, if there exist events $c, d \in S$ such that $\tau(a, b, c, d) = 0$. From the clock axioms it will follow that $\sim_\tau$ is an equivalence relation on S, and the equivalence classes are called *time-slices*.

3.2. **Clock axioms.**

We write down axioms for the map $\tau : S^4 \longrightarrow \mathbb{R} \cup \{\infty\}$. Motivated by our discussion in 2.1, we aim to find an affine line T and a map

$$t : S \longrightarrow T$$

such that

$$t(a) - t(b) = \tau(a, b, c, d)\big(t(c) - t(d)\big),$$

for all $a, b, c, d \in S$ with $c \nsim_\tau d$. These axioms for τ give the basic properties of each observer's clock, and from these properties we will show that we may view *time* as a surjection t from spacetime to an affine line (i.e. our aim is to show that the map τ endows the set T of equivalence classes of $\sim_\tau$ with the structure of an affine line). Of course, there are many ways of axiomatizing the structure of an affine line in terms of the map τ, we just exhibit one such choice.

Axioms 2 (Clock Axioms). *These are the clock axioms.*

CA0: *There exist $a, b \in S$ such that $a \nsim_\tau b$.*

CA1: *For all $a, c, d \in S$ with $c \nsim_\tau d$, we have $\tau(a, a, c, d) = 0$.*

CA2: *For all $a, b, c, d \in S$, we have $\tau(a, b, c, d) = \infty$ if and only if $c \sim_\tau d$.*

CA3: *For all $a, a', b, c, d \in S$ with $c \nsim_\tau d$, we have*

$$a \sim_\tau a' \iff \tau(a, b, c, d) = \tau(a', b, c, d).$$

CA4: *For all $a, b, c, e, f \in S$ with $e \nsim_\tau f$, we have*

$$\tau(a, b, e, f) + \tau(b, c, e, f) + \tau(c, a, e, f) = 0.$$

CA5: *For all $a, b, c, d, e, f \in S$ with $a \nsim_\tau b$, $c \nsim_\tau d$, and $e \nsim_\tau f$, we have*

$$\tau(a, b, c, d)\tau(c, d, e, f)\tau(e, f, a, b) = 1.$$

CA6: *For all $b, c, d \in S$, with $c \nsim_\tau d$, and for all $\lambda \in \mathbb{R}$, there exists $a \in S$ such that*

$$\tau(a, b, c, d) = \lambda.$$

Theorem 3. *If a map $\tau : S^4 \longrightarrow \mathbb{R} \cup \{\infty\}$ satisfies the clock axioms, then there exist an affine line T and a map $t : S \longrightarrow T$, such that*

$$t(a) - t(b) = \tau(a,b,c,d)\big(t(c) - t(d)\big),$$

for all $a, b, c, d \in S$ with $c \nsim_\tau d$.

We give a proof of this theorem in Appendix B.

3.3. **Distance.**

D1 (measurability).: Each observer $o \in O$ is equipped with a *laboratory.*

This means that there is a map

$$\delta_o : S^4 \longrightarrow \mathbb{R}_{\geq 0} \cup \{\infty\},$$

called the *ratio of distances*, and a set $E_o \subseteq S$, called the set of *self-events*, which together satisfy the *rod axioms* below.

D2 (consistency).: Observers agree on the ratios of distances between (not necessarily simultaneous) pairs of simultaneous events.

Remark 4. There is no sense in which we (or our observers) can measure the distance between a pair of non-simultaneous events. It is meaningless to talk of such a measurement. Furthermore, there is no sense in which a subset L of spacetime forms a straightline, unless L lies entirely within one time-slice. *Although there are plenty of isomorphisms between a pair of time-slices, no one is distinguished.*

3.4. **Rod axioms.** We write down axioms for the map $\delta : S^4 \longrightarrow \mathbb{R}_{\geq 0} \cup \{\infty\}$ and set $E \subseteq S$. Our aim is to axiomatise in terms of δ and E the existence of a three-dimensional vector space, D, and a map

$$r : S \longrightarrow D$$

together with a norm[4]

$$\| \cdot \| : D \longrightarrow H,$$

where H is a half-line, such that $\delta(a,b,c,d)$ corresponds to the ratio of $\|r(a) - r(b)\|$ to $\|r(c) - r(d)\|$. See Theorem 10 for more details. These axioms give the basic properties of each observer's laboratory. As above, there are very many ways of axiomatizing the structure of 3-space in terms of the map δ, we just exhibit one such choice.

Definition 5. We say that events $a, b \in S$ are *coincident* (with respect to δ), and write $a \sim_\delta b$, if there exist events $c, d \in S$ such that $\delta(a,b,c,d) = 0$. From the rod axioms, it will follow that $\sim_\delta$ is an equivalence relation on S, and the equivalence classes are called *points*. The set of points $D = D_\delta := S/\sim_\delta$ is called *laboratory space* (with respect to δ). Moreover, it follows that E is a nonempty subset of a single point (i.e. equivalence class), which we denote by e. From the rod axioms it also follows that δ is $\sim_\delta$-invariant, i.e. δ depends only on the $\sim_\delta$-equivalence classes of its arguments. Thus δ induces a map $\breve{\delta}$ on quadruples of elements of D:

$$\breve{\delta} : D^4 \longrightarrow \mathbb{R}_{\geq 0} \cup \{\infty\}.$$

Next, we introduce a variety of geometric relations on laboratory space D. We say that $v \in D$ is *between* $u, w \in D$, and write $\mathbf{betw}(u,v,w)$, if, for some $x, y \in D$ with $x \neq y$, we have

$$\breve{\delta}(u,v,x,y) + \breve{\delta}(v,w,x,y) = \breve{\delta}(u,w,x,y).$$

From the rod axioms it follows that $\mathbf{betw}(u,v,w)$ is equivalent to $\mathbf{betw}(w,v,u)$, and also that $\mathbf{betw}(u,u,v)$ holds, for all $u, v, w \in D$. We say points $u_1, ..., u_n \in D$ are *collinear*, and write $\mathbf{coll}(u_1, \ldots, u_n)$, if at least one of $\mathbf{betw}(v_1,v_2,v_3)$, $\mathbf{betw}(v_2,v_3,v_1)$, or $\mathbf{betw}(v_3,v_1,v_2)$ holds, for every triple $v_1, v_2, v_3 \in \{u_1, \ldots, u_n\}$. Straightaway, it follows that $\mathbf{coll}(u_1, \ldots, u_n)$ does not depend on the order of the u_i, and that $\mathbf{coll}(u,u,v)$ holds for all $u, v \in D$.

[4]See Definition 36.

A *line segment* is a 2-set of points, i.e. $\{u, v\}$, for $u, v \in D$ and $u \neq v$, which we denote by uv for short. Thus $uv = vu$. Two line segments uv, xy are *collinear*, and we write $\mathbf{Coll}(uv; xy)$, if $\mathbf{coll}(u, v, x, y)$ holds. It follows from the axioms that **Coll** defines an equivalence relation on the set of line segments. A *straight line*[5] is an equivalence class of line segments with respect to **Coll**, and the straight line consisting of the equivalence class of uv will be denoted $\underline{uv}$. The set of straight lines is denoted $\mathbb{L}$.

We say points $u_1, \ldots, u_n \in D$ are *coplanar*, and we write $\mathbf{copl}(u_1, \ldots, u_n)$, if there exists a line segment xy such that $\breve{\delta}(u_i, x, x, y) = \breve{\delta}(u_i, y, x, y)$, for every i. A *triangle* is a non-colliner 3-set of points, i.e. $\{u, v, w\}$, for distinct $u, v, w \in D$, which we denote by uvw for short. Immediately it is clear that $\mathbf{copl}(u_1, \ldots, u_n)$ does not depend on the order of the u_i. Also, it follows from the axioms that every triple of points $u_1, u_2, u_3 \in D$ is coplanar. Moreover, **Copl** defines an equivalence relation on triangles. A *plane* is an equivalence class of triangles with respect to coplanarity, and the plane consisting of the equivalence class of uvw will be denoted $\underline{uvw}$. The set of planes is denoted $\mathbb{P}$. We will also need the notion of a *labelled triangle*, which is an ordered triple (u, v, w) of three distinct points. Two labelled triangles (u, v, w) and (u', v', w') are *congruent* if

$$\breve{\delta}(u, v, u', v') = \breve{\delta}(v, w, v', w') = \breve{\delta}(w, u, w', u') = 1.$$

Two straight lines $\underline{vw}$ and $\underline{xy}$ are *parallel*, and we write $\mathbf{parallel}(\underline{vw}, \underline{xy})$, if v, w, x, y are coplanar, and either $\underline{vw}$ and $\underline{xy}$ coincide or do not meet. It follows from the axioms that congruence is an equivalence relation on labelled triangles and that parallelism is an equivalence relation on straight lines.

We can now introduce the incidence relations, as follows. Given a point $x \in D$ and a straight line $\underline{uv}$, we write $x \in \underline{uv}$ to mean that $\mathbf{coll}(u, v, x)$ holds. Similarly, given a plane $\underline{uvw}$, we write $x \in \underline{uvw}$ to mean that $\mathbf{copl}(u, v, w, x)$ holds. We write $\underline{xy} \subset \underline{uvw}$ if

$$(z \in \underline{xy} \implies z \in \underline{uvw}),$$

for all $z \in D$.

Remark 6. We express several of our axioms, specifically **RA6-RA16**, in terms of D and the other geometric notions, for example, straight lines and planes. The well-definedness of each of these depends on the axioms, but there is no circularity. For example, the well-definedness of D depends on **RA0**, **RA1**, and **RA4**. Similarly, straight lines and planes are well-defined by Proposition 52, the proof of which makes use only of **RA8** and **RA10**.

Remark 7. When $\delta = \delta_o$, then D is the laboratory of the observer o, and all of the definitions which we have given in terms of δ may thought of as depending instead on the observer.

Axioms 8 (Rod Axioms). *These are the Rod Axioms.*

RA0: *There exist* $a, b \in S$ *with* $a \not\sim_\delta b$.
RA1: *For all* $a, b, c, d \in S$ *with* $c \not\sim_\delta d$, *we have* $\delta(a, a, c, d) = 0$ *and* $\delta(a, b, c, d) = \delta(b, a, c, d)$.
RA2: *For all* $a, b, c, d \in S$, *we have* $\delta(a, b, c, d) = \infty$ *if and only if* $c \sim_\delta d$.
RA3: *For all* $a, a', b, c, d \in S$ *with* $c \not\sim_\delta d$, *we have*

$$a \sim_\delta a' \iff \delta(a, b, c, d) = \delta(a', b, c, d).$$

RA4: *For all* $a, b, c, e, f \in S$ *with* $e \not\sim_\delta f$, *we have*

$$\delta(a, c, e, f) \leq \delta(a, b, e, f) + \delta(b, c, e, f).$$

RA5: *For all* $a, b, c, d, e, f \in S$ *with* $a \not\sim_\delta b$, $c \not\sim_\delta d$, *and* $e \not\sim_\delta f$, *we have*

$$\delta(a, b, c, d)\delta(c, d, e, f)\delta(e, f, a, b) = 1$$

RA6: *For all* $b, c, d, e \in S$, *with* $c \not\sim_\delta d$, *and for all* $\lambda \in \mathbb{R}_{\geq 0}$, *there exists* $a \in R$ *such that* $\mathbf{coll}(a, b, e)$ *and*

$$\delta(a, b, c, d) = \lambda.$$

[5]We reserve the word 'line' to mean a 1-dimensional vector space, as in Definition 34.

RA7: *The set E is nonempty and, for all $a, b \in E$, we have $a \sim_\delta b$.*
RA8: *Let $u, v, w, x, y, z \in D$. If $w \neq x$ and $\mathbf{coll}(u, v, w, x)$ and $\mathbf{coll}(w, x, y, z)$, then $\mathbf{coll}(u, v, y, z)$.*
RA9: *For all $x, y, z \in D$ we have $\mathbf{copl}(x, y, z)$.*
RA10: *Given three triangles $x_1x_2x_3$, $y_1y_2y_3$, and $z_1z_2z_3$, if both $\mathbf{Copl}(x_1x_2x_3; y_1y_2v_3)$ and $\mathbf{Copl}(y_1y_2y_3; z_1z_2z_3)$, then $\mathbf{Copl}(x_1x_2x_3; z_1z_2z_3)$.*
RA11: *There exist $w, x, y, z \in D$ such that $\mathbf{copl}(w, x, y, z)$ is false.*
RA12: *Let $\underline{x_1x_2x_3}$ be any plane, let $\underline{y_1y_2}$ be any straight line, and suppose that $y_1, y_2 \in \underline{x_1x_2x_3}$. Then $\underline{y_1y_2} \subset \underline{x_1x_2x_3}$.*
RA13: *Let $\underline{x_1x_2}$ be a straight line and let $y_1 \in D$. There is exactly one straight line $\underline{y_1y_2}$ which is parallel to $\underline{x_1x_2}$.*
RA14: *Let $\underline{x_1x_2x_3}$ and $\underline{y_1y_2y_3}$ be two distinct planes, and let z_1 be a point lying on both planes. Then there exits a point $z_2 \neq z_1$ which also lies on both planes.*
RA15: *Let x, y_1, y_2, z_1, z_2 be five distinct points such that $\underline{y_1y_2}$ and $\underline{z_1z_2}$ are parallel, and x lies on both $\underline{y_1z_1}$ and $\underline{y_2z_2}$. Then $\breve{\delta}(x, y_1, x, z_1) = \breve{\delta}(x, y_2, x, z_2)$.*
RA16: *Let (u, v, w) and (u', v', w') be two congruent labelled triangles, and let $x \in \underline{vw}$ and $x' \in \underline{v'w'}$. Suppose that $\breve{\delta}(v, x, w, x) = \breve{\delta}(v', x', w', x')$. Then $\breve{\delta}(u, x, u', x') = 1$.*
RA17: *Let $\underline{uvw}$ be a triangle. There exists $x \in \underline{vw}$ such that there is exactly one $y \in \underline{vw}$ which is not equal to x and for which we have $\breve{\delta}(u, x, v, w) = \breve{\delta}(u, y, v, w)$.*

Definition 9 (See Definition 42)**.** Let H denote a half-line[6]. A 3-*space* (over H) is a three-dimensional real vector space D equipped with a function $\|\cdot\| : D \longrightarrow H$ which satisfies

(1) $\|v\| = 0$ if and only if $v = 0$,
(2) $\|\lambda v\| = |\lambda|\,\|v\|$,
(3) $\|u + v\| \leq \|u\| + \|v\|$, and
(4) $2\|u\|^2 + 2\|v\|^2 = \|u + v\|^2 + \|u - v\|^2$,

for all $u, v \in D$ and $\lambda \in \mathbb{R}$.

Thus a 3-space (over H) is a three-dimensional Euclidean space except that the codomain of the norm is a half line instead of $\mathbb{R}_{\geq 0}$. We note that the fourth axiom, 4., doesn't directly make sense in H, since H is not equipped with multiplicative structure. Nevertheless, we may interpret 4. by evaluating it in $\mathbb{R}_{\geq 0}$, using some identification (by an isomorphism) between the half-lines $\mathbb{R}_{\geq 0}$ and H. Since 4. is an equation of homogeneous polynomials it doesn't matter which identification we choose.

Theorem 10. *If the map $\delta_o : S^4 \longrightarrow \mathbb{R}_{\geq 0} \cup \{\infty\}$ and the set E_o satisfy the rod axioms, then there exist a half-line H_o, a 3-space D_o which includes a norm $\|\cdot\| : D_o \longrightarrow H_o$, and a map $r_o : S \longrightarrow D_o$, such that*

$$||r_o(a) - r_o(b)|| = \delta_o(a, b, c, d)||r_o(c) - r_o(d)||$$

for all $a, b, c, d \in S$ with $c \not\sim_\delta d$.

We give a proof of this theorem in Appendix C.

Definition 11. The direct product $T \times D_o$ is called *time-position space*. The elements of the tuple $(t, x_o) \in T \times D_o$ are called the *time* and *position of an event*.

Each observer has her own time-position space.

3.5. **Handedness.** Let H be a half-line, let D be a 3-space over H. Although in a 3-space we cannot directly speak of an orthonormal basis (since H contains no distinguished non-zero element), we can give the following definition. A triple $(a_1, a_2, a_3) \in D$ is a *relatively orthonormal basis* of D if

(1) $[a_1, a_2] = [a_1, a_3] = [a_2, a_3] = 0$ and
(2) $\|a_1\| = \|a_2\| = \|a_3\| > 0$.

[6] See Definition 36.

When discussing symmetries of 3-spaces and related structures, it is useful to think of a 3-space D (over H) instead as a pair $D = (V, H)$, which is closer to the more formal point-of-view adopted in A. Given two such 3-spaces $D_1 = (V_1, H_1)$ and $D_2 = (V_2, H_2)$, by an *isomorphism* $D_1 \longrightarrow D_2$ we mean a pair $\phi_D = (\phi_V, \phi_H)$ of maps $\phi_V : V_1 \longrightarrow V_2$ and $\phi_H : H_1 \longrightarrow H_2$, such that ϕ_H is an isomorphism of half-lines, ϕ_V is a isomorphism of vector spaces, and $\|\phi_V(x)\| = \phi_H(\|x\|)$, for all $x \in V_1$. An *automorphism* of one such 3-space $D = (V, H)$ is simply an isomorphism $D \longrightarrow D$. Finally, an *isometry* of $D = (V, H)$ is an automorphim such that ϕ_H is the identity map.

The automorphisms of D form a group $\mathrm{Aut}(D)$, and the isometries of D form a subgroup $\mathrm{Isom}(D) \leq \mathrm{Aut}(D)$. In fact $\mathrm{Aut}(D)$ splits into the direct product $\mathbb{R}_{>0} \times \mathrm{Isom}(D)$, where the first factor acts on D by scalar multiplication.[7] Moreover, $\mathrm{Isom}(D)$ is isomorphic to the orthogonal group $O(3, \mathbb{R})$ (viewed as a group of matrices), and the isomorphism is determined by a choice of relatively orthonormal basis. Viewing $\mathrm{Isom}(D)$ as a topological group, the connected component of the identity is a subgroup G_0 of index 2. In fact, under any of the isomorphisms $\mathrm{Isom}(D) \longrightarrow O(3, \mathbb{R})$, i.e. with respect to any choice of relatively orthonormal basis), G_0 is identified with the special orthogonal group $SO(3, \mathbb{R})$ of 'orientation preserving' orthogonal matrices.

The group $\mathrm{Aut}(D)$ acts regularly on the relatively orthonormal bases, and so the subgroup $G_1 := \mathbb{R}_{>0} \times G_0$ also acts on the relatively orthonormal bases of D and has two orbits.

We may extend the notion of 'relatively orthonormality' from laboratory space to spacetime, as follows. Let $o \in O$ be an observer, and let D_o and $r_o : S \longrightarrow D_o$ be the natural map as above. We say that $(a_0, a_1, a_2, a_3) \in S^4$ is a *relatively orthonormal* quadruple of events if $(r_o(a_1) - r_o(a_0), r_o(a_2) - r_o(a_0), r_o(a_3) - r_o(a_0))$ is a relatively orthonormal basis of D_o. For each observer $o \in O$, the *handedness relation* η_o is the 8-ary relation on S which is defined by writing

$$\eta_o(a_0, a_1, a_2, a_3, b_0, b_1, b_2, b_3),$$

if and only if both (a_0, a_1, a_2, a_3) and (b_0, b_1, b_2, b_3) are relatively orthonormal quadruples of events, and $(r_o(a_1)-r_o(a_0), r_o(a_2)-r_o(a_0), r_o(a_3)-r_o(a_0))$ and $(r_o(b_1)-r_o(b_0), r_o(b_2)-r_o(b_0), r_o(b_3)-r_o(b_0))$ are in the same orbit under the action of G_1.

It follows immediately that η_o is an equivalence relation on relatively orthonormal quadruples of events.

H1 (consistency).: Observers agree of the relative handedness of two (not necessarily simultaneous) ordered quadruples of simultaneous events.

That is, for any two observers $o, o' \in O$ and any eight events $a_0, a_1, a_2, a_3, b_0, b_1, b_2, b_3 \in S$, if a_0, a_1, a_2, a_3 are pairwise simultaneous and if b_0, b_1, b_2, b_3 are pairwise simultaneous then

$$\eta_o(a_0, a_1, a_2, a_3, b_0, b_1, b_2, b_3)$$

holds if and only if

$$\eta_{o'}(a_0, a_1, a_2, a_3, b_0, b_1, b_2, b_3)$$

holds.

3.6. **Symmetry.** The symmetries of Spacetime are defined by the following two axioms.

T3 (homogeneity and isotropy).: An observer cannot measure *absolute time*, nor can she measure *absolute time differences*, except time difference zero, nor can she measure a preferred direction of time.

There is no distinguished point in time. Nor is there a distinguished time difference. Nor is there a distinguished direction in time.

[7] The action by scalars is diagonal in the sense that $\lambda \in \mathbb{R}^\times$ corresponds to the automorphism (ϕ_V, ϕ_H), where ϕ_V is scalar multiplication of V by λ and ϕ_H is scalar multiplication of H by $|\lambda|$.

D3 (homogeneity and isotropy).: An observer cannot measure *absolute position*, nor can she measure *absolute distance*, except distance zero, nor can she measure *absolute orientation.*

Using the above axioms, we are able to give a description of spacetime as a structure $\mathbf{S}$, rather than simply as a set S. The axioms T1 and T2 assert that $\mathbf{S}$ is equipped with a map satisfying the assumptions of Theorem 3, which in turn implies that $\mathbf{S}$ is equipped with an affine line $\mathbf{T}$ and a map $\tau' : \mathbf{S} \longrightarrow \mathbf{T}$. Likewise, the axioms D1 and D2 together with Theorem 10, assert that spacetime $\mathbf{S}$ is equipped with a family of 3-spaces $\mathbf{D}_o$ and a family of maps $\delta'_o : \mathbf{S} \longrightarrow \mathbf{D}_o$, one for each observer o.

3.7. **Transformations between laboratories.** Given two affine 3-spaces $E_1 = (A_1, V_1, H_1)$ and $E_2 = (A_2, V_2, H_2)$, by an *affine isomorphism* $E_1 \longrightarrow E_2$ we mean a triple $\phi_E = (\phi_A, \phi_V, \phi_H)$ such that (ϕ_V, ϕ_H) is an isomorphism $(V_1, H_1) \longrightarrow (V_2, H_2)$ of 3-spaces, and $\phi_A : A_1 \longrightarrow A_2$ is such that

$$\phi_A(a +_\sigma v) = \phi_A(a) +_\sigma \phi_V(v),$$

for all $a \in A_1$ and $v \in V_1$. An *affine automorphism* of one such affine 3-space $E = (A, V, H)$ is simply an affine isomorphism $E \longrightarrow E$. Finally, an *affine isometry* of $E = (A, V, H)$ is an affine automorphism such that ϕ_H is the identity map.

The affine automorphisms of E form a group $\mathrm{AAut}(E)$, and the affine isometries of E form a subgroup $\mathrm{AIsom}(E) \leq \mathrm{AAut}(E)$. In fact $\mathrm{AAut}(E)$ is the semidirect product $V \rtimes \mathrm{Aut}(V)$, and correspondingly $\mathrm{AIsom}(A)$ is the semidirect product $V \rtimes \mathrm{Isom}(V)$, where in each case the first factor acts on A by affine translation.

Given any 3-space $D = (V, H)$, we may straightforwardly form an affine 3-space $E = (A, V, H)$ by taking $A = V$. This amounts to forgetting the origin.

Let $o, o' \in O$ be two observers with laboratories $D_o = (V_o, H_o)$ and $D_{o'} = (V_{o'}, H_{o'})$, associated affine 3-spaces $E_o = (A_o, V_o, H_o)$ and $E_{o'} = (A_{o'}, V_{o'}, H_{o'})$, and maps $r_o : S \longrightarrow D_o$ (over H_o) and $r_{o'} : S \longrightarrow D_{o'}$ (over $H_{o'}$). Since observers agree on simultaneity, by T2, and on the coincidence of simultaneous events, by D2, for each $t_0 \in T$ there is a bijection

$$C_o^{o'}[t_0] : D_o \longrightarrow D_{o'}$$

such that

$$C_o^{o'}[t_0](r_o(a)) = r_{o'}(a),$$

for all events $a \in S$ with $t(a) = t_0$. In fact, also by D2, the maps $C_o^{o'}[t_0]$ are in fact affine isomorphisms $E_o \longrightarrow E_{o'}$. For $t_0, t_1 \in T$, the composition

$$C_o^{o'}[t_1, t_0] := C_{o'}^{o}[t_1] \circ C_o^{o'}[t_0]$$

is an affine *isometry* of E_o. Indeed, for fixed t_0, the map

$$\begin{aligned} T &\longrightarrow \mathrm{AIsom}(E_o) \\ t_1 &\longmapsto C_o^{o'}[t_1, t_0] \end{aligned}$$

describes a family of affine isometries of E_o, parameterized by $t_1 \in T$. Under the assumption of twice-differentiability, see Definition 18, to such a family of affine isometries we may assign a velocity $\dot{X} : T \longrightarrow D_o$ and an angular velocity $\Omega : T \longrightarrow D_o$, as well as their associated accelerations.

3.8. **The Structure of Spacetime.** We summarize the discussion of the previous sections in the following corollary.

Corollary 12. *The mathematical structure of classical spacetime is described by the tuple* $(S, O, T, t, (D_o)_{o \in O}, (r_o)_{o \in O})$.

4. Mechanical systems

Thus far we have considered the measurements we may make of spacetime and the structure these impose on it. Spacetime is the arena in which mechanics takes place. We now consider mechanics itself.

4.1. **Worldlines.** In this section we introduce worldlines, which allow us to discuss the motions of bodies.

Definition 13. A *worldline* is a section of the surjection $\tau' : S \longrightarrow T$, i.e. a map

$$w : T \longrightarrow S$$

such that $\tau' \circ w$ is the identity.

Let o be an observer. Then for each $t \in T$ there is a unique event $a \in S$ such that

$$\tau'(a) = t \text{ and } \delta'_o(a) = 0.$$

This allows us to associate a worldline $w[o] : T \longrightarrow S$ to each observer, as follows.

Definition 14. For $t \in T$, we let $w[o](t) = a$ where a is the unique event as above. We call $w[o]$ the *worldline associated with the observer* o.

4.2. **Motions.** Observers perceive worldlines as *motions*, which we describe in the next definition. We denote by V_T the vector space (which is a line) underlying T, and we denote by D_o the 3-space corresponding to the laboratory of an observer $o \in O$.

Definition 15. Let $o \in O$ be an observer. A *motion* is a map $x : T \longrightarrow D_o$. Such a motion is *linear* if its graph

$$\text{graph}(x) = \{(t, x(t)) \in T \times D_o \mid t \in T\}$$

is a (straight) line in the time-position space $T \times D_o$ of o. With respect to o, for each worldline $w : T \longrightarrow S$, there is an *induced motion*

$$x_o^w : T \longrightarrow D_o,$$

which is defined by $x_o^w := r_o \circ w$.

Definition 16. Two observers $o, o' \in O$ are *relatively inertial* if, for all worldlines $w : T \longrightarrow S$, the induced motion x_o^w is linear if and only if the induced motion $x_{o'}^w$ is linear.

Roughly speaking, two observers are relatively inertial if, with respect to each other, they are undergoing uniform motion without rotation. More precisely, o and o' are relatively inertial if and only if, for some (equivalently, all) t_0, the map $t_1 \longmapsto C_o^{o'}[t_1, t_0]$ is of the form $t_1 \longmapsto [x \longmapsto x +_\sigma (t_1 - t_0)v]$, for some $v \in V_o$.

4.3. **Derivatives of motions.** Using 'little-o' notation, as explained in A.7, we may introduce the derivative of a motion.

Definition 17. Let $x : T \longrightarrow D_o$ and let $t \in T$. We say that x is *differentiable* at t if there exists a linear map $dx[t] : V_T \longrightarrow D_o$ such that

$$||x(t + h) - x(t) - dx[t](h)|| = o(|h|).$$

Now, suppose that we fix a *standard* non-zero time interval $s \in V_T \setminus \{0\}$. Using s as a scale, we may recast the derivative $dx[t]$ of the motion x as a motion:

$$\begin{aligned} \dot{x} : T &\longrightarrow D_o \\ t &\longmapsto dx[t](s). \end{aligned}$$

This recasting makes use of the evaluation map

$$dx[t] \longmapsto dx[t](s).$$

Continuing to view s as fixed, and thus the derivative of x as a motion, it makes sense to ask when $\dot{x}$ is differentiable. If it is, then we denote its derivative by

$$\ddot{x} : T \longrightarrow D_o.$$

Again, we stress that both $\dot{x}$ and $\ddot{x}$ depend on our choice of scale s. A clock T together with a choice of a standard time interval s is said to be *dimensionalized.*

4.4. **Mechanical systems.** In this section we study the motion of systems of bodies, via the formalism of *mechanical systems* which we introduce in the following definition.

Definition 18. A *mechanical system* is a pair $(o, \mathbf{w})$, where o is an observer and $\mathbf{w}$ is a set of worldlines, such that for each $w \in \mathbf{w}$, the induced motion $x_o^w : T \longrightarrow D_o$ is twice differentiable.

Note that D_o is a 3-space, and so there is a meaning to saying that a motion $x : T \longrightarrow D_o$ is twice differentiable.

M1 (consistency).: Let o, o' be two observers and $\mathbf{w}$ a set of worldlines such that $(o, \mathbf{w})$ is a mechanical system. Then $(o', \mathbf{w})$ is also a mechanical system.

Remark 19. In our approach, we have not endowed spacetime itself with a differential structure, or even a topological structure. Rather, axiom M1 imposes a restriction on the set of observers. Essentially, it says that two observers cannot disagree about when the induced motions of a worldline are twice differentiable.

Definition 20. Suppose that we fix a *standard* non-zero time difference $s \in V_T \setminus \{0\}$. With respect to s, we may define a *metric* on T as follows:

$$\begin{aligned} d_s : T \times T &\longrightarrow \mathbb{R}_{\geq 0} \\ (t_1, t_2) &\longmapsto \sqrt{|\tau(t_1, t_2, t_1, t_1 + s)|}. \end{aligned}$$

A clock together with a choice of standard time difference is said to be *dimensionalized.*

Definition 21. Suppose that we fix a *standard* non-zero rod $r \in D_o \setminus \{e_o\}$. With respect to r, we may define a *norm* on D_o as follows:

$$\begin{aligned} || \cdot ||_r : D_o &\longrightarrow \mathbb{R}_{\geq 0} \\ u &\longmapsto \sqrt{d(u, e_o, r, e_o)}. \end{aligned}$$

A laboratory together with a choice of a standard rod is said to be *dimensionalized.*

In the following three definitions, $\mathbf{w} = (w_i)_i$ is an indexed set of worldlines. With respect to an observer o, for brevity we denote the motion induced by w_i by $x_o^i := x_o^{w_i}$. Moreover, we write $\mathbf{x}_o = (x_o^i)_i$. We continue to work with a dimensionalized laboratory D_o with standard rod $r \in D_o \setminus \{e_o\}$, as well as a dimensionalized clock T with standard time interval $s \in V_T \setminus \{0\}$.

We will also consider functions $\mathbf{F} : D_o^{2n} \times T \longrightarrow D_o^n$, which we view as tuples $\mathbf{F} = (F^i)$ of functions $F^i : D_o^{2n} \times T \longrightarrow D_o$.

Definition 22. We say that $\mathbf{F} : D_o^{2n} \times T \longrightarrow D_o^n$ *describes* a mechanical system $(o, \mathbf{w})$ if

$$\ddot{\mathbf{x}}_o(t) = \mathbf{F}(\mathbf{x}_o(t), \dot{\mathbf{x}}_o(t), t),$$

for all $t \in T$.

If a mechanical system $(o, \mathbf{w})$ is described by $\mathbf{F}$, and o' is another observer, then

$$\begin{aligned} \mathbf{F}' : D_{o'}^{2n} \times T &\longrightarrow D_{o'}^n \\ (\mathbf{x}, \mathbf{y}, t) &\longmapsto C_o^{o'}[t]\big(\mathbf{F}(C_{o'}^o[t](\mathbf{x}, \mathbf{y}), t)\big) \end{aligned}$$

describes $(o', \mathbf{w})$.

Such a function $\mathbf{F}$, with any choice of initial conditions, together determine the system $(o, \mathbf{w})$, as is standard with a system of ordinary second-order differential equations. Each mechanical system is described by some such function $\mathbf{F}$. The mechanical system $(o, \mathbf{w})$ is *trivially* described by the function

$$\begin{aligned} \mathbf{F} : D_o^{2n} \times T &\longrightarrow D_o^n \\ (\mathbf{x}_o(t), \dot{\mathbf{x}}_o(t), t) &\longmapsto \ddot{\mathbf{x}}_o(t). \end{aligned}$$

The accelerations $\ddot{\mathbf{x}}_o$ are already functions of time. So the *existence* of such a function is a vacuous requirement on a mechanical system. Instead, classical mechanics studies those force laws that satisfy two requirements: Galilean invariance and the pairwise anti-symmetry described by G3.

Definition 23. A mechanical system $(o, \mathbf{w})$ is *Galilean* if it is described by a function $\mathbf{F}$ that is invariant under the action of the Galilean group. See for example Arnold [5, Chapter 1.2].

Proposition 24. If a mechanical system $(o, \mathbf{w})$, which is described by $\mathbf{F}$, is Galilean, then $\mathbf{F}$ is a function of the separation and relative velocities of pairs of motions only. More precisely, $(o, \mathbf{w})$ is Galilean if there exists

$$\mathbf{F} : D_o^{2n} \longrightarrow D_o^n$$

such that

$$\ddot{x}_o^i(t) = F^i(x_o^1(t) - x_o^i(t), ..., x_o^n(t) - x_o^i(t), \dot{x}_o^1(t) - \dot{x}_o^i(t), ..., \dot{x}_o^n(t) - \dot{x}_o^i(t)),$$

for all $t \in T$ and all i.

Proof. See Arnold [5, Chapter 1.2]. □

Definition 25. A mechanical system $(o, \mathbf{w})$ is *Newtonian* if there exist functions

$$F^{i,j} : D_o^4 \times T \longrightarrow D_o,$$

for each pair (i, j) with $i \neq j$, and there exist *masses* $m_i \in \mathbb{R}_{>0}$, for each i, such that

$$m_i \ddot{x}_o^i(t) = \sum_{j \neq i} F^{i,j}(x_o^i(t), x_o^j(t), \dot{x}_o^i(t), \dot{x}_o^j(t), t)$$

and moreover

$$F^{i,j}(x_o^i(t), x_o^j(t), \dot{x}_o^i(t), \dot{x}_o^j(t), t) = -F^{j,i}(x_o^j(t), x_o^i(t), \dot{x}_o^j(t), \dot{x}_o^i(t), t),$$

for all $t \in T$ and all indices i, j with $i \neq j$. In this case we say that the elements of $(m_i)_i$ are the *masses* of this system, and that o is an *inertial observer* of this system. We call the first of the above equations *Newton's equation*. The function $F^{i,j}$ is known as the *force acting on i due to its interaction with j*. The sum over all such functions (i.e. the right-hand side of Newton's equation) is known as the *total force acting on i*.

Remark 26. Consider a mechanical system $(o, \mathbf{w})$ where $\mathbf{w} = (w_1)$ is a singleton, which we will call 'the worldline of an isolated body'. The above definition of 'Newtonian' amounts to requiring that there exists $m_1 \in \mathbb{R}_{>0}$ such that

$$m_1 \ddot{x}_o^1(t) = 0,$$

for all t. Thus, $(o, \mathbf{w})$ is Newtonian if and only if x_1 is linear. Note also that in this case 'there exists $m_1 \in \mathbb{R}_{>0}$' is equivalent to 'for all $m_1 \in \mathbb{R}_{>0}$'.

Remark 27. Although we use the word 'inertial' here, our usage is very different from that in G1. Our definition of an inertial reference has more in common with that given by Einstein [7], who wrote that an inertial reference frame is one in which 'physical laws hold good in their simplest form'. We call the function $F^{i,j}$ 'the force acting on i due to its interaction with j'. This is simply a name. Force is neither a primitive quantity nor a synonym for the product of the mass and acceleration.

Proposition 28. Let $(o, \mathbf{w})$ be a Galilean and Newtonian mechanical system, described by $(F^{i,j})_{i,j}$, as in Definition 25, and let o' be another observer whose laboratory has acceleration $\ddot{X}$ and angular velocity Ω with respect to o. Then

$$m_i \ddot{x}^i_{o'}(t) = \sum_{j \neq i} F^{i,j}(x^i_{o'}(t), x^j_{o'}(t), \dot{x}^i_{o'}(t), \dot{x}^j_{o'}(t), t) - m_i \ddot{X}(t) - m_i \Omega(t) \times (\Omega(t) \times x^i_{o'}(t))$$
$$-2m_i \Omega(t) \times \dot{x}^i_{o'}(t) - m_i \dot{\Omega}(t) \times x^i_{o'}(t).$$

Proof. A proof is given by Arnold [5, Chapter 6]. □

The testable claim in classical mechanics is that for *given* F, $\ddot{X}$, and Ω a *given* mechanical system satisfies the equation in Proposition 28. An observer is thus testing a claim about particular set of worldlines as viewed by an inertial observer *together with* a claim about her motion relative to that inertial observer. The observer must find $\ddot{X}$ and Ω informally, or make an approximation to them. It might be the case, for example, that $\ddot{X}$ and Ω are approximately constant over some timescale of interest to the observer.

We might consider, but reject, the following axiom.

M2: For a given set of worldlines $(\mathbf{w})$ there exists a superset $\mathbf{w}' \supseteq \mathbf{w}$ and an observer o' such that $(o', \mathbf{w}')$ is Newtonian and Galilean.

This axiom claims that at least one *closed* system exists, as does an inertial observer of that closed system. But such a claim is of a very different nature from our axioms concerning the characteristic of clocks and laboratories (T1–3, D1–3). The difference is the axioms' testability. We may test a given system to see if it satisfies the axioms of clocks and laboratories. But, given a clock and laboratory it is unclear how to test the claim made by M2. Suppose there exists a set $\mathbf{w}$ of worldlines which does not satisfy M2. We might perpetually hunt through pairs of observers and supersets of worldlines without ever being able to definitively say that none satisfies the claim.

Proposition 29. Let $(o, \mathbf{w})$ be a Galilean and Newtonian mechanical system. Let o' be another observer and suppose that o and o' are relatively inertial. Then $(o', \mathbf{w})$ is a Galilean and Newtonian mechanical system.

Proof. This follows immediately from Proposition 28. □

Example 30. A mechanical system $(o, \mathbf{w})$ is *gravitational* if the worldlines in $\mathbf{w}$ are disjoint and there exists a tuple $\tilde{\mathbf{m}} = (\tilde{m}_i)$ of positive real numbers such that for each i the following equation holds for all $t \in T$:

$$m_i \ddot{x}^i_o(t) = \sum_{j \neq i} \frac{\tilde{m}_i \tilde{m}_j (x^j_o(t) - x^i_o(t))}{||x^j_o(t) - x^i_o(t)||^3_r}.$$

In this case we say that the elements of $\tilde{\mathbf{m}}$ are the *gravitational masses* of the system.

Remark 31. It seems surprising that the above equation, in particular the tuple $\mathbf{m}$, depends on a choice of dimensionalization. This is a consequence of the definition of the derivative of a motion (Definition 17).

Of the four known fundamental interactions (gravitation, electromagnetism, the weak interaction, and the strong interaction) classical mechanics treats only one.[8] Yet classical mechanics has great success in accounting for non-gravitational systems that properly belong to other domains of physics (relativistic mechanics, quantum mechanics, or quantum field theory). This fact is remarkable. Consider, for example, an elastic system, the pairwise interactions of which

[8] In an electromagnetic system of n bodies, the acceleration of the ith body is $\ddot{x}_i(t) = q_i (E + \dot{x}_i \times B) / m_i$, where the electric field E and magnetic field B are given by Maxwell's equations. The right-hand side of this equation is a function of the ith body's velocity, $\dot{x}_i$, and not its relative velocity with respect to the other bodies of the system. Therefore it is not Galilean invariant.

are successfully described by Hooke's law, $m_i\ddot{x}_i = -k_{ij}(x_j - x_i)$, even though the system is governed by atomic interactions within a spring. Indeed classical mechanics allows for counterfactual mechanical systems. For example, it is perfectly classical to consider the Solar System as governed by an inverse-cube law of gravity. In this respect classical mechanics is a sand-box theory that allows us to consider a toy system described by arbitrary F so long as it remains Newtonian and Galilean.

5. Conclusion

We have described our formalization of classical mechanics as *operational* or *instrumental.* It is instrumental insofar as it concerns the instruments we use to perform physical experiments, and operational insofar as it concerns the practical procedures whereby we perform such experiments. In experiments pertaining to mechanics, we use clocks and laboratories to measure the positions and times of events. We have therefore given axiomatized definitions of clocks (CA1–7) and laboratories (RA1–10, HA1–7). We have then given axioms concerning the behaviour of these clocks and laboratories for different observers (T1–3, D1–3). From these we have deduced a structure for spacetime.

In attempting this formalization, we had two questions in mind. What is an inertial reference frame? What is a force? Neither appears in our formalization in their conventional sense. To say that there exist inertial reference frames, in which a body on which no force acts is not accelerated, is to say that there is an observer-independent definition of a straight line in spacetime. But there is no operational method for identifying such a straight line. As such they cannot be said to exist. Spacetime consists of a series of time-slices (i.e. sets of simultaneous events) but there is no canonical isomorphism between these time-slices. Instead, each observer provides her own isomorphism. Similarly, there is no method for directly measuring a force. Positions (or, rather, separations) are measured by rods. Times (or, rather, time differences) are measured by clocks. But there are no instruments that measure forces. Any force meter (such as a spring balance) measures separations and is calibrated using a theory of force (such as Hooke's law). Forces are the purported cause of accelerations. But we lose nothing (indeed we gain in parsimony) if we think in terms of accelerations rather than forces. Forces are conceptually superfluous. In fact, we do not even measure accelerations directly (although we may measure arbitrarily good approximations of accelerations by measuring changes in positions over time). Instead, we can only hypothesize that the position of the i-th body in a system satisfies some given second-order differential equation, of the form $m_i\ddot{x}^i_o(t) = F^i(\mathbf{x}_o(t), \dot{\mathbf{x}}_o(t), t)$. In our formalism we call F^i the force acting on the i-th body, but it is neither a primitive quantity nor a mere synonym for the product of a body's mass and acceleration. If, for a particular observer, the force is a sum over antisymmetric pairwise interactions between bodies we say that observer is inertial. But our use of the term 'inertial' is very different from that of convention. In its conventional sense, an inertial frame may not be identified independently of a force. In our sense an inertial frame has no meaning independently of a force. Our formalism does not require the existence of inertial frames. Their existence is a matter for experiment. The testable claim in classical mechanics is that a given system of worldlines obeys a given second order differential equation for a given observer. This requires us to define an equivalent Newtonian and Galilean system, and to make a claim about the observer's motion relative to one such system. Claims about Newtonian and Galilean systems in themselves are not testable. Instead, we can only test a compound claim.

We have not said precisely how this test is to be performed. We leave this for future work, anticipating that we should treat measurements of positions and times as the realizations of random variables, and use the equation of motion to construct a statistical model which we may then fit to observations. All constants that appear in the force law (mass, gravitational mass, elastic constant, etc.) are then parameters of this model. We also plan to axiomatize special relativity in a similar way, anticipating that we will need at least to modify the way in which clocks and labs are understood to be consistent, and the symmetries we require of mechanical systems. Such an axiomatization of special relativity will allow us to explore the the essential similarities of, and differences between, classical and relativistic mechanics.

Appendix A. Mathematical Details

A.1. **Standard Mathematical Structures.** Most of the Mathematical structures used in this paper are familiar from Euclidean Geometry and Linear Algebra. We refer the reader to Roe, [8], for a more thorough introduction to the subject. For convenience, we briefly recall the main definitions below. All of our structures are defined over the real numbers $\mathbb{R}$, so we say a 'vector space' rather than a 'real vector space', for example. For simplicity of notation, we use the same symbol in different contexts, for example we denote addition by '+', and scalar multiplication by 'μ', in almost every structure. For further brevity, we often write $\alpha v = \mu(\alpha, v)$.

Definition 32. A *vector space* is a tuple $\mathbf{V} = (V; +, -, 0, \mu)$, where $(V; +, -, 0)$ is an abelian group and $\mu : \mathbb{R} \times V \longrightarrow V$ is called 'scalar multiplication', which together satisfy the following axioms. For all $u, v \in V$ and $\alpha, \beta \in \mathbb{R}$ we have:

(1) $\alpha(u + v) = \alpha u + \alpha v$,
(2) $(\alpha + \beta)u = \alpha u + \beta v$,
(3) $(\alpha\beta)u = \alpha(\beta u)$, and
(4) $1u = u$.

The above axioms amount to saying that $\alpha \longmapsto [u \longmapsto \alpha u]$ is an embedding $\mathbb{R} \longrightarrow \mathrm{End}(V, +, -, 0)$ of rings with unity.

Definition 33. An *affine space* is a triple $\mathbf{A} = (A, \mathbf{V}; \sigma)$, where $\mathbf{V} = (V; +, -, 0)$ is a vector space, A is a set, and $\sigma : V \times A \longrightarrow A$ is called the 'translation map', which together satisfy the following axioms, where we write $v +_\sigma a := \sigma(v, a)$. For all $u, v \in V$ and $a, b \in A$ we have:

(1) $(u + v) +_\sigma a = u +_\sigma (v +_\sigma a)$,
(2) $0 +_\sigma a = a$, and
(3) there exists a unique $w \in V$ such that $w +_\sigma a = b$.

Unless there is a risk of ambiguity, we write such an affine space simply as $\mathbf{A} = (A, \mathbf{V})$.

Affine spaces can equally well be axiomatised in terms of a binary operation $-_\sigma : A \times A \longrightarrow V$, related to $+_\sigma$ by:

$$a_1 -_\sigma a_2 = v \quad \text{if and only if} \quad v +_\sigma a_2 = a_1,$$

for all $a_1, a_2 \in A$ and $v \in V$.

Definition 34. A *line* is a one-dimensional vector space.

Definition 35. An *affine line* is an affine space $\mathbf{A} = (A, \mathbf{V})$ such that $\mathbf{V}$ is a line.

A.2. **Half-lines.** We make several uses of the notion of a 'half-line'. Although it is a very simple idea, for the lack of a convenient reference we give an axiomatic definition.

Definition 36. A *half-line* is a tuple $\mathbf{H} = (H; +, 0, \mu)$, where $+ : H \times H \longrightarrow H$, $0 \in H$, and $\mu : \mathbb{R}_{\geq 0} \times H \longrightarrow H$ is called 'scalar multiplication', which together satisfy the following axioms. For all $u, v, w \in H$ and all $\alpha, \beta \in \mathbb{R}_{\geq 0}$ we have:

(1) $u + (v + w) = (u + v) + w$
(2) $u + 0 = 0 + u = u$
(3) $u + v = v + u$
(4) $u + w = v + w \implies u = v$
(5) $\alpha(u + v) = \alpha u + \alpha v$,
(6) $(\alpha + \beta)u = \alpha u + \beta v$,
(7) $(\alpha\beta)u = \alpha(\beta u)$, and
(8) $1u = u$.

(9) if $u \neq 0$ then there exists unique $\gamma \in \mathbb{R}_{\geq 0}$ such that $\gamma u = v$.

There is a natural total order induced on a half-line $\mathbf{H}$ by the action of $\mathbb{R}_{\geq 0}$. Namely, for $x, y \in H$, we write $x \leq y$ if and only if either $x = 0$ or $\gamma x = y$, for some $\gamma \in \mathbb{R}_{\geq 0}$ with $\gamma \geq 1$. Then $\leq$ is a total order on $\mathbf{H}$ which is compatible with addition, in that:

$$x \leq y \implies x + z \leq y + z,$$

for all $x, y, z \in H$. Furthermore, for $x, y \in H$ with $x \leq y$, we write

$$y - x := \begin{cases} (\gamma - 1)x & \text{if } \gamma x = y \text{, where } \gamma \geq 1, \\ y & \text{if } x = 0. \end{cases}$$

Naturally we can view $\mathbb{R}_{\geq 0}$ as a half-line in its own right.

Definition 37. Let $\mathbf{H}_1, \mathbf{H}_2$ be half-lines. A map $\phi : H_1 \longrightarrow H_2$ is a *homomorphism* of half-lines if, for all $u, v \in H_1$ and $\alpha, \beta \in \mathbb{R}_{\geq 0}$, we have

$$\phi(\alpha u + \beta v) = \alpha\phi(u) + \beta\phi(v).$$

In this case we write $\phi : \mathbf{H}_1 \longrightarrow \mathbf{H}_2$. An *isomorphism* of half-lines is a bijective homomorphism of half-lines.

Remark 38. Foer each half-line $\mathbf{H}$ and each $v \in H$ there exists a unique homomorphism $\phi_v : \mathbb{R}_{\geq 0} \longrightarrow \mathbf{H}$ of half-lines such that $\phi_v(1) = v$. Moreover, if $v \neq 0$ then ϕ_v is an isomorphism.

A.3. **Directed lines.**

Definition 39. A *directed line* is a pair $\vec{\mathbf{L}} = (\mathbf{L}; P)$ of a line $\mathbf{L}$ together with a subset $P \subset L$ which forms a half-line, when equipped with the structure induced from $\mathbf{L}$.

A directed line is equipped with a total ordering, compatible with the line structure, as follows. For $u, v \in L$ we write $u \leq v$ to mean that $v - u \in P$. Then $\leq$ is a total ordering on L and for $u, v, w \in L$ and $\alpha \in \mathbb{R}_{\geq 0}$ we have

(1) $u \leq v \implies u + w \leq v + w$ and
(2) $u \leq v \implies \alpha u \leq \alpha v$.

Given a directed line $\vec{\mathbf{L}}$, we denote by $\vec{\mathbf{L}}_{\geq 0}$ the half-line structures induced on P.

A.4. **Halving and doubling.** We discuss the relationships between lines, directed lines, and half-lines. Let $\mathbf{L}$ be a line. The action of $\mathbb{R}_{>0}$ partitions L into three orbits, namely the singleton $\{0\}$ and two sets P_1, P_2. For $i \in \{1, 2\}$, we write $\vec{\mathbf{L}}_i := (\mathbf{L}, P_i \cup \{0\})$. Then $\vec{\mathbf{L}}_1$ and $\vec{\mathbf{L}}_2$ are distinct directed lines, which are permuted by an automorphism of $\mathbf{L}$, namely the map $x \longmapsto -x$. Thus we may not canonically identify one over the other.

For $x, y \in L$, we write $|x| = |y|$ if $x = y$ or $x + y = 0$. This defines an equivalence relation on L. Let $|x|$ denote the equivalence class of x, and let $L_\pm$ denote the set of equivalence classes. For $x, y \in L$ we write

$$|x| \oplus |y| := \begin{cases} |y| & \text{if } x = 0, \\ |x + y| & \text{if } y = \alpha x \text{, for some } \alpha \in \mathbb{R}_{\geq 0}, \\ |x - y| & \text{else,} \end{cases}$$

which defines an addition $\oplus : L_\pm \times L_\pm \longrightarrow L_\pm$. Also, for $x \in L$ and $\alpha \in \mathbb{R}_{\geq 0}$ we write

$$\alpha|x| = |x\alpha|,$$

which defines a scalar multiplication $\mu_\pm : \mathbb{R}_{\geq 0} \times L_\pm \longrightarrow L_\pm$. It is routine to check that these maps are well-defined. We refer to $\mathbf{L}_\pm = (L_\pm; \oplus, |0|, \mu_\pm)$ as the *halving* of $\mathbf{L}$. Indeed $\mathbf{L}_\pm$ is a half-line.

Remark 40. Let $\mathbf{L}$ be a line. Then its halving $\mathbf{L}_\pm$ is a half-line.

Now we go in the other direction: we build a directed line from a half-line as follows. Let $\mathbf{H} = (H; +, 0, \mu)$ be a half-line. We define the equivalence relation $\sim_\tau$ on $H \times H$ by writing $(x_1, y_1) \sim_\tau (x_2, y_2)$ to mean that $x_1 + y_2 = x_2 + y_1$. It can be checked that $\sim_\tau$ really is an equivalence relation. Write $[x, y]$ for the equivalence class of (x, y), and let $\hat{H} := (H \times H)/\sim_\tau$ be the set of equivalence classes. We define addition $\hat{+} : \hat{H} \times \hat{H} \longrightarrow \hat{H}$ by setting:

$$[x_1, y_1]\hat{+}[x_2, y_2] := [x_1 + x_2, y_1 + y_2].$$

It can be checked that $\hat{+}$ is well-defined. We also define the map $\hat{\mu} : \mathbb{R} \times \hat{H} \longrightarrow \hat{H}$, which we write as $\alpha(x, y) := \hat{\mu}(\alpha, (x, y))$, by setting

$$\alpha(x, y) := (\alpha x, \alpha y).$$

Finally, we write $\hat{0} := [0, 0]$. We refer to $\hat{\mathbf{H}} = (\hat{H}; \hat{+}, \hat{0}, \hat{\mu})$ as the *double* of $\mathbf{H}$. We also write $P_1 := \{[x, 0] \in \hat{H} \mid x \in H\}$ and $P_2 := \{[0, y] \in \hat{H} \mid y \in H\}$, and we refer to $\vec{\mathbf{H}}^1 := (\hat{H}; P_1)$ and $\vec{\mathbf{H}}^2 := (\hat{H}; P_2)$ as the *directed doubles* of $\mathbf{H}$.

Remark 41. Let $\mathbf{H}$ be a half-line. Then its double $\hat{\mathbf{H}}$ is a line and both its directed doubles $\vec{\mathbf{H}}^1$ and $\vec{\mathbf{H}}^2$ are directed lines.

A.5. **3-space.** By definition, a norm on a vector space $\mathbf{V} = (V; +, -, 0, \mu)$ is a map

$$\| \cdot \| : V \longrightarrow \mathbb{R}_{\geq 0},$$

which satisfies

(1) $\|v\| = 0$ if and only if $v = 0$,
(2) $\|\lambda v\| = |\lambda| \, \|v\|$, and
(3) $\|u + v\| \leq \|u\| + \|v\|$,

for all $u, v \in D$ and $\lambda \in \mathbb{R}$. We introduce an analogue of this notion in which we 'forget' the multiplicative structure of the codomain, and only retain the structure of a directed line (in the case of an inner product) or a half-line (in the case of a norm).

Definition 42 (See Definition 9)**.** A 3-*space* is a tuple $\mathbf{D} = (\mathbf{V}, \mathbf{H}; \| \cdot \|)$, where $\mathbf{V}$ is a three-dimensional vector space, $\mathbf{H}$ is a half-line, and $\| \cdot \| : V \longrightarrow H$ satisfies

(1) $\|v\| = 0$ if and only if $v = 0$,
(2) $\|\lambda v\| = |\lambda| \, \|v\|$,
(3) $\|u + v\| \leq \|u\| + \|v\|$, and
(4) $2\|u\|^2 + 2\|v\|^2 = \|u + v\|^2 + \|u - v\|^2$,

for all $u, v \in V$ and $\lambda \in \mathbb{R}$. We call the map $\| \cdot \|$ a *norm*, although this again is a slight abuse of notation.

As remarked above, the fourth axiom, 4., doesn't directly make sense in $\mathbf{H}$, since $\mathbf{H}$ has no multiplication. Instead, we interpret 4. by evaluating it in $\mathbb{R}_{\geq 0}$, using some identification (via an isomorphism) between the half-lines $\mathbb{R}_{\geq 0}$ and $\mathbf{H}$. Since the equation in 4. is homogeneous it doesn't matter which identification we choose. Under each of these identifications, 4. is the *polarization identity*, which in turn means that $\| \cdot \|$ corresponds to an inner product, we may be recovered by writing

$$[u, v] := \frac{1}{4}\big(\|u + v\|^2 - \|u - v\|^2\big).$$

A different choice of identification between $\mathbb{R}_{\geq 0}$ and $\mathbf{H}$ will correspond to a different inner product, and this correspondence is compatible with the scalar multiplication in an appropriate sense. Thus, if $\mathbf{D} = (\mathbf{V}, \mathbf{H}; \| \cdot \|)$ is a 3-space, then there is a directed line $\vec{\mathbf{L}} = (\mathbf{L}; P)$ and a map $[\ ,\ ] : V \times V \longrightarrow L$ which is positive-definite, symmetric, and bilinear. We call the map $[\ ,\ ]$ an *inner product*, although this is a very slight abuse of language, as explained above. There is an identification

$$H \longrightarrow P$$
$$\|v\| \longmapsto [v, v],$$

which is certainly not linear, but is homogeneous of degree 2.

Note that the usual notion of Euclidean 3-space is recovered by adjoining to a 3-space a constant non-zero element of $\mathbf{H}$. This is equivalent to distinguishing an isomorphism of half-lines $\mathbf{H} \longrightarrow \mathbb{R}_{\geq 0}$.

Finally, an *affine 3-space* is a tuple $\mathbf{A} = (A, \mathbf{V}, \mathbf{H}; \sigma, \|\cdot\|)$, where $(A, \mathbf{V}; \sigma)$ is an affine space and $(\mathbf{V}, \mathbf{H}; \|\cdot\|)$ is a 3-space. Moreover, given any 3-space $\mathbf{D} = (\mathbf{V}, \mathbf{H}; \|\cdot\|)$, we may straightforwardly form the affine 3-space $\mathbf{E} = (A, \mathbf{V}, \mathbf{H}; \sigma, \|\cdot\|)$ by taking $A = V$, and defining σ using addition in $\mathbf{V}$. This amounts to forgetting the origin.

A.6. **Orthogonality.** Let $\mathbf{D} = (\mathbf{V}, \mathbf{H}; \|\cdot\|)$ be a 3-space. For each $h \in H_{>0}$, there is an isomorphism of half-lines

$$\mathbf{H} \longrightarrow \mathbb{R}_{\geq 0}$$

defined by $h \longmapsto 1$. Taking such an isomorphism as an identification, we may square a norm, thus it makes sense to write

$$\|w\|^2 = \|u\|^2 + \|v\|^2,$$

for $u, v, w \in D$. Since this equality is equivalent to the vanishing of a homogeneous polynomial, if it holds for one such identification, then it holds for all such. Thus we may write $[u, v] = 0$ to mean that $\|u + v\|^2 = \|u\|^2 + \|v\|^2$ holds, for some (equivalently, for all) such identifications. The relation '$[u, v] = 0$' then satisfies

(1) $[u, u] = 0$ if and only if $\|u\| = 0$, and
(2) $[u, v] = 0$ if and only if $[v, u] = 0$,

for all $u, v \in V$.

A.7. **Little-o notation.** Let $\mathbf{D} = (\mathbf{V}, \mathbf{H}; \|\cdot\|)$ be a 3-space. Fix an isomorphism $\phi : \mathbf{H}_{\pm} \longrightarrow \vec{\mathbf{L}}_{\geq 0}$ of half-lines.

Definition 43. Let $f : H \longrightarrow V$ be a function from the underlying set of $\mathbf{H}$ to the underlying set of $\mathbf{V}$. We write

$$||f(h)|| = o_{\phi}(h), \quad \text{as } h \longrightarrow 0,$$

if for all $\varepsilon \in \mathbb{R}_{>0}$ there exists $\delta \in H_{\pm} \setminus \{0\}$ such that for all $h \in H$ we have

$$0 < |h| < \delta \implies ||f(h)|| < \phi(\varepsilon |h|).$$

Lemma 44. *The relation '$||f(h)|| = o_{\phi}(h)$' does not depend on the choice of isomorphism ϕ of half-lines.*

Proof. Let $\phi, \psi : \mathbf{H}_{\pm} \longrightarrow \vec{\mathbf{L}}_{\geq 0}$ be two isomorphisms of half-lines. Then $\phi^{-1} \circ \psi : \mathbf{H}_{\pm} \longrightarrow \mathbf{H}_{\pm}$ is an isomorphism. Therefore there exists $\eta \in \mathbb{R}_{>0}$ such that for all $h \in H$ we have

$$\phi^{-1} \circ \psi(|h|) = \eta |h|.$$

Suppose that $||f(h)|| = o_{\phi}(h)$ holds. Let $\varepsilon \in \mathbb{R}_{>0}$. Applying the assumption not to ε but to $\eta\varepsilon$, there exists $\delta^{\phi}_{\eta\varepsilon} \in H_{\pm} \setminus \{0\}$ such that for all $h \in H$ we have

$$0 < |h| < \delta^{\phi}_{\eta\varepsilon} \implies ||f(h)|| < \phi(\eta\varepsilon|h|) = \psi(\varepsilon|h|).$$

Therefore choosing $\delta^{\psi}_{\varepsilon} := \delta^{\phi}_{\eta\varepsilon}$ shows that $||f(h)|| = o_{\psi}(h)$ holds, as required. □

Therefore we are able to write $||f(h)|| = o(h)$ to unambiguously mean that for some/all isomorphisms $\phi : \mathbf{H}_{\pm} \longrightarrow \vec{\mathbf{L}}_{\geq 0}$ we have $||f(h)|| = o_{\phi}(h)$.

Appendix B. A clock as an affine line

We explore consequences of the clock axioms. We are given a map $\tau : S^4 \longrightarrow \mathbb{R} \cup \{\infty\}$ which satisfies the clock axioms **CA0-CA6**. as detailed above in 3.2. Recall the following definition.

Definition 45 (cf Definition 1)**.** We say that events $a, b \in S$ are *simultaneous*, and write $a \sim_\tau b$, if there exist events $c, d \in S$ such that $\tau(a, b, c, d) = 0$.

It follows from the axioms that simultaneity $\sim_\tau$ is an equivalence relation on S. We give some basic algebraic properties of τ which are routine consequences of the clock axioms.

Lemma 46. *Let $a, b, c, d, e, f \in S$.*

(i) If $e \nsim_\tau f$ then $\tau(a, b, e, f) + \tau(b, a, e, f) = 0$.
(ii) If $a \nsim_\tau b$ then $\tau(a, b, a, b) = 1$.
(iii) If $a \nsim_\tau b$ then $\tau(a, b, b, a) = -1$.
(iv) If $a \nsim_\tau b$ and $c \nsim_\tau d$ then $\tau(a, b, c, d)\tau(c, d, a, b) = 1$.
(v) If $c \nsim_\tau d$ and $e \nsim_\tau f$ then $\tau(a, b, c, d)\tau(c, d, e, f) = \tau(a, b, e, f)$.
(vi) If $c \nsim_\tau d$ then $\tau(a, b, c, d) = \tau(b, a, d, c)$.

Furthermore, τ is $\sim_\tau$-invariant, i.e. invariant under simultaneity, as in the following proposition.

Proposition 47. Let $a, a', b, b', c, c', d, d' \in S$ with $a \sim_\tau a'$, $b \sim_\tau b'$, $c \sim_\tau c'$, and $d \sim_\tau d'$. Then

$$\tau(a, b, c, d) = \tau(a', b', c', d').$$

Let $\tilde{a}$ denote the equivalence class of a with respect to $\sim_\tau$, i.e. the set of events simultaneous with a, and let $T := S/\sim_\tau$ denote the set of equivalence classes. We have the natural map

$$\begin{aligned} t : S &\longrightarrow T \\ a &\longmapsto \tilde{a}. \end{aligned}$$

Since by Proposition 47 τ is $\sim_\tau$-invariant, τ induces a map

$$\begin{aligned} \tilde{\tau} : T^4 &\longrightarrow \mathbb{R} \cup \{\infty\} \\ (\tilde{a}, \tilde{b}, \tilde{c}, \tilde{d}) &\longmapsto \tau(a, b, c, d). \end{aligned}$$

Our task is to describe the algebraic structure induced on T by the map $\tilde{\tau}$.

Proposition 48. For all $b, c \in S$ with $b \nsim_\tau c$ the map

$$\begin{aligned} \phi_{b,c} : T &\longrightarrow \mathbb{R} \\ \tilde{a} &\longmapsto \tau(a, b, c, b) \end{aligned}$$

is a well-defined bijection.

Pulling back the structure of $\mathbb{R}$ via the map $\phi_{b,c}$ endows T with the structure of a one-dimensional real vector space, i.e. a line. This vector space structure does not depend on c, as long as b and c are not simultaneous, and only depends on the $\sim_\tau$-equivalence class of b. We denote this vector space by $\mathbf{T}_b = (T\ ;\ 0_b, +_b, (\lambda_b)_{\lambda \in \mathbb{R}})$, where $+_b$ is the addition, λ_b is scalar multiplication by λ, and $0_b = \tilde{b}$ is the zero. The addition and scalar multiplication work as follows:

$$\lambda_b \tilde{a}_1 +_b \mu_b \tilde{a}_2 = \tilde{a}_3$$

if and only if

$$\lambda \tau(a_1, b, c, b) + \mu \tau(a_2, b, c, b) = \tau(a_3, b, c, b),$$

for *some* (equivalently, *all*) $c \in S$ with $c \nsim_\tau b$.

For each b, $\mathbf{T}_b$ is a line, and $(T, \mathbf{T}_b)$ is an affine line. It remains to identify a canonical copy of the lines $\mathbf{T}_b$.

Definition 49. We define the binary relation $\approx$ on the set $T^2 = T \times T$ by

$$(\tilde{a}, \tilde{b}) \approx (\tilde{c}, \tilde{d}) \iff \tau(a,b,c,d) = 1 \text{ or } (a \sim_\tau b \text{ and } c \sim_\tau d).$$

The two cases in the definition are mutually exclusive, since $c \sim_\tau d$ implies $\tau(a,b,c,d) = \infty \neq 1$. The relation $\approx$ well-defined, and is an equivalence relation on T^2. The set of equivalence classes $T_\approx := T^2/\approx$ inherits the structure of a line, denoted $\mathbf{T}_\approx$, which acts on T as follows. For $v \in T_\approx$ and $a \in T$ we write $v +_\sigma a = b$ to mean that $(a,b) \approx v$. Therefore $(T, \mathbf{T}_\approx; \sigma)$ is an affine line, establishing Theorem 3.

Appendix C. A laboratory as a 3-space

We explore consequences of the Rod Axioms. We are given a set S and a map

$$\delta : S^4 \longrightarrow \mathbb{R}_{\geq 0} \cup \{\infty\},$$

which we suppose satisfies the Rod Axioms **RA0**-**RA16**, as detailed above in 3.4. We aim to give a proof of Theorem 10. In this case, unlike that of Clocks, there is a suitable reference, namely [8], and we have chosen our axioms deliberately to mirror those found there.

Recall that above we introduced the relations of *coincidence*, *collinearity*, *coplanarity*, and others. Of course, these relations depended entirely on the observer o. In this section, however, we will ignore observers, since we prove Theorem 10 for a given observer. It follows from the axioms that coincidence $\sim_\delta$ is an equivalence relation. We also have the following straightforward algebraic properties of δ.

Lemma 50. *Let $a,b,c,d,e,f \in S$.*

(i) If $a \not\sim_\delta b$ then $\delta(a,b,a,b) = 1$.
(ii) If $a \not\sim_\delta b$ then $\delta(a,b,b,a) = 1$.
(iii) If $a \not\sim_\delta b$ and $c \not\sim_\delta d$ then $\delta(a,b,c,d)\delta(c,d,a,b) = 1$.

Let $\breve{a}$ denote the equivalence class of a, i.e. the set of events coincident with a, and let $D := S/\sim_\delta$ denote the set of equivalence classes. Then we have the natural map

$$r : S \longrightarrow D$$
$$a \longmapsto \breve{a}.$$

Let o denote the equivalence class which includes E, which lies in a single equivalence class by **RA7**. It follows from the axioms (principally from **RA3**) that δ is $\sim_\delta$-invariant. Thus δ induces a map

$$\breve{\delta} : D^4 \longrightarrow \mathbb{R}_{\geq 0} \cup \{\infty\}.$$

We aim to describe the structure induced by $\breve{\delta}$ on D, and on its quotients.

Remark 51. Our aim is to show that, if (S, δ) satisfies the Rod Axioms, then Roe's 'Axioms of Incidence' hold for the incidence structure $(D, \mathbb{L}, \mathbb{P})$ consisting of the set D equipped with the set of straight lines $\mathbb{L}$ and the set of planes $\mathbb{P}$, together with the incidence relations described above.

The following proposition is straightforward.

Proposition 52. **Coll** is an equivalence relation on 2-sets of points. and **Copl** is an equivalence relation on non-collinear 3-set of points.

Given a straight line $\underline{uv}$ (or a plane $\underline{uvw}$), it is natural to consider the set of points x lying on that straight line (or plane). This set is the *extension* of the straight line (or plane), and we may fail to distinguish between the two, unless it would cause confusion.

Proposition 53 (Modularity, cf [8, Axiom 6])**.** Let $\underline{x_1x_2x_3}$ and $\underline{y_1y_2y_3}$ be two distinct planes, and let z_1 be a point lying on both planes. Then there is exactly one line $\underline{z_1z_2}$ which lies in both planes. I.e. the intersection of the extensions of the two planes is the extension of a line.

Proof. Existence is **RA14** and **RA12**. For uniqueness, we apply **RA10**. □

This verifies Roe's Axiom 6. Roe's Axiom 4 is our **RA12**, and Roe's first two axioms are verified by the following proposition.

Proposition 54 (cf [8, Axioms 1,2])**.** Through any two distinct points there is exactly one straight line. Through any three non-collinear points there is exactly one plane.

Proof. If u, v are distinct then they lie on $\underline{uv}$. If u, v lie on $\underline{xy}$ then $\mathbf{coll}(u, x, y)$ and $\mathbf{coll}(v, x, y)$. From **RA8** it follows that $\mathbf{coll}(u, v, x, y)$, i.e. $\underline{uv} = \underline{xy}$.

Suppose that u, v, w are three non-collinear points. Then they lie on the plane $\underline{uvw}$. Suppose that they also lie on the plane $\underline{xyz}$. Then $\underline{uv}, \underline{vw}, \underline{uw}$ are three distinct lines which, by **RA12**, lie in $\underline{xyz}$. From Modularity it follows that $\underline{uvw} = \underline{xyz}$. □

Proposition 55 (cf [8, Axiom 3])**.** Any line contains at least two distinct points. Any plane contains at least two distinct lines. There are at least two distinct planes.

Proof. The first statement follows from the definition of a line. For the second statement: consider the plane $\underline{uvw}$. By assumption, u, v, w are non-collinear. By **RA12**, $\underline{uv}, \underline{vw}, \underline{uw} \subseteq \underline{uvw}$. The final claim follows from **RA11**. □

Proposition 56. Each straight line inherits from $\breve{\delta}$ the structure of an affine line.

Proof. This follows from the rod axioms (restricted to a straight line), just as Theorem 3 follows from the clock axioms. □

We claim that, in effect, this proposition verifies Roe's Axiom 7 and Axiom 8. To see this, we consider a straight line $\underline{ab}$ on which lie two distinct points a and b. Roe's axioms require there is a ruler (in his terminology) on $\underline{ab}$ which assigns a to 0 and b to 1. Such a ruler on $\underline{ab}$ is provided by the map

$$R_{a,b} : c \longmapsto \begin{cases} -\delta(a, c, a, b) & \text{if } \mathbf{betw}(c, a, b), \\ \delta(a, c, a, b) & \text{otherwise.} \end{cases}$$

The family $\{R_{a,c} \mid c \in \underline{ab}, c \neq a\}$ of these rulers, for fixed a, together with a zero element, forms a line (in the sense of Definition 34). The identification of $R_{a,b}$ with $R_{a',b'}$ whenever $\delta(a', b', a, b) = 1$ results in a line. This line then acts regularly on the straight line $\underline{ab}$, which of course matches the structure of an affine line on $\underline{ab}$, Definition 34.

Finally, we have straightforwardly included Roe's Axiom 5, Axiom 9, Axiom 11, and Axiom 12 as **RA13**, **RA15**, **RA16**, and **RA17**.

Theorem 57. *We have the following*

(1) $\mathbf{L} := (L; +, 0, (\lambda)_{\lambda \in \mathbb{R}_{\geq 0}})$ *is a half-line.*
(2) $\mathbf{D} := (D, L; +, 0, (\lambda)_{\lambda \in \mathbb{R}_{\geq 0}}, [\,,\,])$ *is a* 3*-space, i.e. a three-dimensional vector space with a positive definite, symmetric, and bilinear form* $[\,,\,] : D^2 \longrightarrow L$*.*

Proof. We have shown that $(D, \mathbb{L}, \mathbb{P})$, together with the incidence relations, satisfies Roe's axioms. From these axioms, Roe establishes the theorem in Chapters 1 and 3 of [8]. □

Moreover we have

$$\delta(a, b, c, d)||\breve{c} - \breve{d}|| = ||\breve{a} - \breve{d}||,$$

for all $a, b, c, d \in S$ with $c \not\sim_\delta d$. Equivalently we have

$$\breve{\delta}(u, v, w, x)||w - x|| = ||u - v||,$$

for all $u, v, w, x \in R$ with $w \neq x$.

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